

# The Weyl-Wigner-Moyal Formalism for Spin

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(Dated: October 16, 2012)

## Abstract

The Weyl-Wigner-Moyal formalism is developed for spin by means of a correspondence between spherical harmonics and spherical harmonic tensor operators. The analogue of the Moyal expansion is developed for the Weyl symbol of the product of two operators in terms of the symbols for the individual operators, and it is shown that in the classical limit, the Weyl symbol for a commutator equals  $i$  times the Poisson bracket of the corresponding Weyl symbols. It is also found that, to the same order, there is no correction in the symbol for the anticommutator.

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For a particle with linear momentum  $p$  and coordinate  $q$ , the Weyl-Wigner-Moyal formalism [1–3] leads to a fascinating correspondence between classical and quantum mechanics, as it provides a mapping between quantum mechanical operators and classical dynamical variables defined as functions on phase space. When the operator is the density matrix, the phase space function is known as the Wigner function. Because this function fails to be nonnegative in all circumstances, it cannot be taken to be a true probability distribution. So while it does nothing to solve the knotty problem of interpreting quantum mechanics, this formalism is the closest one can get to turning quantum mechanics into a purely classical statistical theory, and it also finds practical application, especially in quantum optics.

It is desirable to have a similar formalism for spin degrees of freedom both for intrinsic reasons, and for developing the semiclassical limit. In their low energy states, many atoms, molecules, and molecular ions behave as particles with a fixed magnitude of spin that is often large, in which case a semiclassical approach is natural. We have found, for example, that the Weyl representation is advantageous in constructing the spin-coherent-state path integral [4]. The problem of developing such a formalism has been approached by several workers over the years [5–9] with varying emphases and methods. The problem is solved in a formal sense and its abstract mathematical aspects are quite highly developed. At the same time, unlike the case of linear position and momentum degrees of freedom, there are few results in closed form or of practical value. To our knowledge, it has not even been shown that the leading classical behavior of the Moyal bracket as the spin magnitude  $j \rightarrow \infty$  is the same as the Poisson bracket. This result is of course to be expected on physical grounds, but that is not the same as actually proving it, and the proof is an important physical check on the correctness of the formal structure. One can further expect higher order corrections in powers of  $1/j$  to aid in developing quantum corrections. While our eventual goal is to address this practical problem, in this paper we only present general results. Our approach is more succinct in some ways than the formally oriented ones, relies more on analysis than algebra, and may appeal to physicists for these reasons.

That a Weyl representation for spin operators has been known for several decades may surprise some readers, as it surprised us. We have found modern day papers which state the contrary, which along with our own ignorance illustrates the surprisingly common occurrence in modern day physics of the same facts being well known to one subcommunity of physicists and totally unknown to others. Therefore, we cannot hope too much that our paper will

lessen this disconnect.

Our basic working tool is the set of spherical harmonic tensor operators  $\mathcal{Y}_{\ell m}(\mathbf{J})$ . We find that the P, Q, and Weyl representations (indicated by the superscripts P, Q, and W) of  $\mathcal{Y}_{\ell m}(\mathbf{J})$  are all proportional to  $Y_{\ell m}(\hat{\mathbf{n}})$ . That is, for a particle of spin  $j$ ,

$$\Phi_{\ell m}^{P,Q,W}(\hat{\mathbf{n}}) = a_{j\ell}^{P,Q,W} Y_{\ell m}(\hat{\mathbf{n}}), \quad (1)$$

where the coefficients are given by

$$a_{j\ell}^P = (j+1)(j+\frac{3}{2})(j+2)\cdots(j+\frac{1}{2}\ell+\frac{1}{2}), \quad (2)$$

$$a_{j\ell}^Q = j(j-\frac{1}{2})(j-1)\cdots(j-\frac{1}{2}\ell+\frac{1}{2}), \quad (3)$$

$$a_{j\ell}^W = (a_{j\ell}^P a_{j\ell}^Q)^{1/2} = \prod_{k=1}^{\ell} ((j+\frac{1}{2})^2 - \frac{1}{4}k^2)^{1/2}. \quad (4)$$

Since any operator can be uniquely and prescriptively expressed as a linear combination of the  $\mathcal{Y}_{\ell m}(\mathbf{J})$ 's, these results constitute an algorithmic solution to the problem of finding the P, Q, and Weyl representations for an arbitrary operator.

Before proving these assertions, we review some basic definitions and known results. The objects of interest are operators for a particle of spin  $j$ , which are functions of  $\mathbf{J}$ , the vector spin operator, with components  $J_x$ ,  $J_y$ , and  $J_z$  that obey the usual commutation rules (setting  $\hbar$  to 1)  $[J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma}J_\gamma$ , and  $\mathbf{J} \cdot \mathbf{J} = J_x^2 + J_y^2 + J_z^2 = j(j+1)$ . For an arbitrary such operator  $A(\mathbf{J})$ , the P and Q symbols are defined by

$$\Phi_A^Q(\hat{\mathbf{n}}) = \langle \hat{\mathbf{n}} | A(\mathbf{J}) | \hat{\mathbf{n}} \rangle, \quad (5)$$

$$A(\mathbf{J}) = \frac{2j+1}{4\pi} \int d\hat{\mathbf{n}} |\hat{\mathbf{n}} \rangle \Phi_A^P(\hat{\mathbf{n}}) \langle \hat{\mathbf{n}}|. \quad (6)$$

Here,  $|\hat{\mathbf{n}} \rangle$  is the (normalized with  $\langle \hat{\mathbf{n}} | \hat{\mathbf{n}} \rangle = 1$ ) spin state with maximum spin projection along the direction  $\hat{\mathbf{n}}$ . That is,  $\mathbf{J} \cdot \hat{\mathbf{n}} |\hat{\mathbf{n}} \rangle = j |\hat{\mathbf{n}} \rangle$ . In particular, with  $|j, m \rangle$  denoting the simultaneous eigenstate of  $\mathbf{J} \cdot \mathbf{J}$  and  $J_z$ ,  $|\hat{\mathbf{z}} \rangle = |j, j \rangle$ . In Eq. (6), the integral is over all  $\hat{\mathbf{n}}$ . It also pays to employ stereographic coordinates on the sphere. If  $\theta$  and  $\varphi$  denote the usual spherical polar coordinates of  $\hat{\mathbf{n}}$ , and we define  $z = \tan \frac{1}{2}\theta e^{i\varphi}$ , then it is a standard result that the (unnormalized) state  $|z \rangle$  defined by

$$|z \rangle = e^{zJ_-} |j, j \rangle = \sum_{m=-j}^j \binom{2j}{j-m}^{1/2} z^{j-m} |j, m \rangle \quad (7)$$

is on the same ray in Hilbert space as  $|\hat{\mathbf{n}}\rangle$ . That is,  $|z\rangle = (1 + |z|^2)^j |\hat{\mathbf{n}}\rangle$ , where the normalization follows from Eq. (7). When  $A = \mathcal{Y}_{\ell m}(\mathbf{J})$ , we write simply  $\Phi_{\ell m}^{P,Q,W}$  for the symbols.

We also recall that Eq. (6) does not define  $\Phi_A^P$  uniquely. The reason is that the matrix elements of  $|\hat{\mathbf{n}}\rangle\langle\hat{\mathbf{n}}|$  in the  $|j, m\rangle$  basis are linear combinations of  $Y_{\ell m}(\hat{\mathbf{n}})$  with  $\ell \leq 2j$ . Because of the orthogonality of the  $Y_{\ell m}$ 's, we may add to  $\Phi_A^P$  any linear combination of  $Y_{\ell m}$ 's with  $\ell > 2j$  without affecting the integral over  $\hat{\mathbf{n}}$ . In this paper we shall fix  $\Phi_A^P$  by demanding that the coefficients of  $Y_{\ell m}$  with  $\ell > 2j$  vanish.

Any physically meaningful phase space representation must be linear, real for Hermitean operators, equal to the number 1 for the identity operator, and covariant under rotations. That is, if the symbol for an operator  $A(\mathbf{J})$  is  $\Phi_A(\hat{\mathbf{n}})$ , and if a rotation takes  $\mathbf{J}$  into  $\mathbf{J}'$  and  $\hat{\mathbf{n}}$  into  $\hat{\mathbf{n}}'$ , then the symbol for  $A(\mathbf{J}')$  must be  $\Phi_A(\hat{\mathbf{n}}')$ . These properties are obeyed by the P and Q symbols (modulo the nonuniqueness of the former). The Weyl symbol must also obey the key extra demand of *traciality* [5]. That is, if  $A$  and  $B$  are two operators, we must have

$$\frac{1}{2j+1} \text{tr}(AB) = \frac{1}{4\pi} \int d\hat{\mathbf{n}} \Phi_A^W(\hat{\mathbf{n}}) \Phi_B^W(\hat{\mathbf{n}}). \quad (8)$$

The left side of this equation may be seen as a quantum mechanical average, and the right side as a classical one over phase space. We denote these as  $\langle \rangle_{\text{qm}}$  and  $\langle \rangle_{\hat{\mathbf{n}}}$ . This condition determines  $\Phi_{\ell m}^W(\hat{\mathbf{n}})$  unambiguously. Further, if  $B$ , say, is taken as the density matrix, then  $\Phi_B^W(\hat{\mathbf{n}})$  is the Wigner function for the system.

Next, we must define the operators  $\mathcal{Y}_{\ell m}(\mathbf{J})$ . We do this in parallel with the spherical harmonics  $Y_{\ell m}(\hat{\mathbf{n}})$  via the Herglotz generating function [10] for the latter:

$$e^{\zeta \mathbf{a} \cdot \mathbf{r}} = \sum_{\ell, m} \sqrt{\frac{4\pi}{2\ell+1}} \frac{r^\ell \zeta^\ell \lambda^m}{\sqrt{(\ell+m)!(\ell-m)!}} Y_{\ell m}(\hat{\mathbf{n}}), \quad (9)$$

$$e^{\zeta \mathbf{a} \cdot \mathbf{J}} = \sum_{\ell, m} \sqrt{\frac{4\pi}{2\ell+1}} \frac{\zeta^\ell \lambda^m}{\sqrt{(\ell+m)!(\ell-m)!}} \mathcal{Y}_{\ell m}(\mathbf{J}). \quad (10)$$

Here,

$$\mathbf{a} = \hat{\mathbf{z}} - \frac{\lambda}{2}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + \frac{1}{2\lambda}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}), \quad (11)$$

with  $\lambda$  and  $\zeta$  being real. The essential property of  $\mathbf{a}$  is that  $\mathbf{a} \cdot \mathbf{a} = 0$ . Further,  $r = |\mathbf{r}|$ , and  $\hat{\mathbf{n}} = \mathbf{r}/r$ . Equation (9) implies the normalization that the integral of  $|Y_{\ell m}|^2$  over all  $\hat{\mathbf{n}}$  is unity.

Equations (9) and (10) show immediately that various relationships among the  $Y_{\ell m}$ 's have direct analogs for the  $\mathcal{Y}_{\ell m}$ 's. We have, for example,

$$Y_{\ell m}^*(\hat{\mathbf{n}}) = (-1)^m Y_{\ell, -m}(\hat{\mathbf{n}}), \quad \mathcal{Y}_{\ell m}^\dagger(\mathbf{J}) = (-1)^m \mathcal{Y}_{\ell, -m}(\mathbf{J}). \quad (12)$$

Further, it is apparent that under rotations  $Y_{\ell m}(\hat{\mathbf{n}})$  and  $\mathcal{Y}_{\ell m}(\mathbf{J})$  transform identically.

We now note that  $\mathcal{Y}_{\ell m}(\mathbf{J})$  is the operator analogue not of the *surface* harmonic  $Y_{\ell m}(\hat{\mathbf{n}})$ , but of the *solid* harmonic  $r^\ell Y_{\ell m}(\hat{\mathbf{n}})$  which has the extra factor  $r^\ell$ . Hence, we expect all three symbols for  $\mathcal{Y}_{\ell m}(\mathbf{J})$  to asymptote to  $j^\ell Y_{\ell m}(\hat{\mathbf{n}})$  as  $j \rightarrow \infty$ . Second, for spin  $j$ , any operator is equivalent to a matrix of order  $2j + 1$ . A Hermitean matrix of this order has  $(2j + 1)^2$  independent parameters, so there can be at most  $(2j + 1)^2$  independent  $\mathcal{Y}_{\ell m}$ 's. This number is exhausted by taking  $\ell$  up to  $2j$ , and the rotational properties then guarantee that  $\mathcal{Y}_{\ell m}(\mathbf{J}) = 0$  if  $\ell > 2j$ .

We now find the various  $\Phi_{\ell m}^Q$ 's. For  $\Phi_{\ell m}^Q$  we take the expectation value of Eq. (9) in the state  $|\hat{\mathbf{n}}\rangle$  to obtain

$$\langle \hat{\mathbf{n}} | e^{\zeta \mathbf{a} \cdot \mathbf{J}} | \hat{\mathbf{n}} \rangle = \sum_{\ell, m} \sqrt{\frac{4\pi}{2\ell + 1}} \frac{\zeta^\ell \lambda^m}{\sqrt{(\ell + m)!(\ell - m)!}} \Phi_{\ell m}^Q(\hat{\mathbf{n}}). \quad (13)$$

We now evaluate the left hand side using the stereographically labeled states  $\{|z\rangle\}$ . Thus,

$$\begin{aligned} \langle \hat{\mathbf{n}} | e^{\zeta \mathbf{a} \cdot \mathbf{J}} | \hat{\mathbf{n}} \rangle &= (1 + |z|^2)^{-2j} \langle z | e^{\zeta \mathbf{a} \cdot \mathbf{J}} | z \rangle \\ &= (1 + |z|^2)^{-2j} \langle j, j | e^{z^* J_+} e^{\zeta \mathbf{a} \cdot \mathbf{J}} e^{z J_-} | j, j \rangle. \end{aligned} \quad (14)$$

The Lie algebra of the  $J_\alpha$ 's allows us to write  $e^{z^* J_+} e^{\zeta \mathbf{a} \cdot \mathbf{J}} e^{z J_-} = e^{u_- J_-} e^{\beta J_z} e^{u_+ J_+}$ , where  $u_\pm$  and  $\beta$  are functions of  $\zeta$ ,  $z^*$ ,  $z$ , and  $\mathbf{a}$ . We thus obtain  $\langle z | e^{\zeta \mathbf{a} \cdot \mathbf{J}} | z \rangle = e^{\beta j}$ , and  $\langle \hat{\mathbf{n}} | e^{\zeta \mathbf{a} \cdot \mathbf{J}} | \hat{\mathbf{n}} \rangle = e^\gamma$ , where  $e^\gamma = e^\beta / (1 + |z|^2)^2$ .

To find  $\gamma$ , we exploit the faithfulness of the  $j = 1/2$  representation of  $\text{SU}(2)$ . In this case,  $(\mathbf{a} \cdot \mathbf{J})^2 = a_i a_j (\delta_{ij} + 2i \epsilon_{ijk} J_k) / 4 = 0$ , so,  $e^{\zeta \mathbf{a} \cdot \mathbf{J}} = 1 + \zeta \mathbf{a} \cdot \mathbf{J}$ . Now,  $\langle \hat{\mathbf{n}} | \mathbf{J} | \hat{\mathbf{n}} \rangle = j \hat{\mathbf{n}}$ , so for  $j = 1/2$ ,  $\langle \hat{\mathbf{n}} | e^{\zeta \mathbf{a} \cdot \mathbf{J}} | \hat{\mathbf{n}} \rangle = 1 + \frac{1}{2} \zeta \mathbf{a} \cdot \hat{\mathbf{n}} = e^{\gamma/2}$ . Hence, for general  $j$ , Eq. (13) may be written as

$$\langle \hat{\mathbf{n}} | e^{\zeta \mathbf{a} \cdot \mathbf{J}} | \hat{\mathbf{n}} \rangle = \left(1 + \frac{1}{2} \zeta \mathbf{a} \cdot \hat{\mathbf{n}}\right)^{2j} \quad (15)$$

$$\begin{aligned} &= \sum_{\ell=0}^{2j} \binom{2j}{\ell} \frac{\zeta^\ell}{2^\ell} (\mathbf{a} \cdot \hat{\mathbf{n}})^\ell \\ &= \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} \binom{2j}{\ell} \frac{1}{2^\ell} \frac{\ell!}{\sqrt{(\ell + m)!(\ell - m)!}} \sqrt{\frac{4\pi}{2\ell + 1}} \zeta^\ell \lambda^m Y_{\ell m}(\hat{\mathbf{n}}), \end{aligned} \quad (16)$$

where in the last line we have used the  $\zeta^\ell$  term in Eq. (9). Comparing with Eq. (13), and simplifying, we obtain

$$\Phi_{\ell m}^Q(\hat{\mathbf{n}}) = \frac{1}{2^\ell} \frac{(2j)!}{(2j-\ell)!} Y_{\ell m}(\hat{\mathbf{n}}) \quad (\ell \leq 2j), \quad (17)$$

which is the same as Eq. (3). As expected,  $\Phi_{\ell m}^Q(\hat{\mathbf{n}})$  is proportional to  $Y_{\ell m}(\hat{\mathbf{n}})$ , and it vanishes if  $\ell > 2j$ . Since any operator is determined by just its diagonal matrix elements in the spin coherent states, we have rededuced that  $\mathcal{Y}_{\ell m} = 0$  for  $\ell > 2j$ . Alternatively, knowing this result, we could have anticipated Eq. (17) from the connection between solid and surface harmonics which ensures that  $\Phi_{\ell m}^Q(\hat{\mathbf{n}}) \sim j^\ell Y_{\ell m}(\hat{\mathbf{n}})$ .

Knowing  $\Phi_{\ell m}^Q(\hat{\mathbf{n}})$ ,  $\Phi_{\ell m}^P$  is simple to find. Taking the expectation value of Eq. (6) in the state  $|\hat{\mathbf{n}}\rangle$ , and noting that  $|\langle \hat{\mathbf{n}} | \hat{\mathbf{n}}' \rangle|^2 = [(1 + \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')/2]^{2j}$ , we obtain

$$\Phi_A^Q(\hat{\mathbf{n}}) = \frac{2j+1}{4\pi} \int d\hat{\mathbf{n}}' \left( \frac{1 + \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'}{2} \right)^{2j} \Phi_A^P(\hat{\mathbf{n}}'). \quad (18)$$

It is straightforward to show that

$$\left( \frac{1 + \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'}{2} \right)^{2j} = \sum_{\ell=0}^{2j} K_{j\ell} P_\ell(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'), \quad (19)$$

where

$$K_{j\ell} = (2\ell+1) \frac{[(2j)!]^2}{(2j-\ell)!(2j+\ell+1)!}, \quad (20)$$

and  $P_\ell$  is the Legendre polynomial of order  $\ell$ . We take  $A = \mathcal{Y}_{\ell m}(\mathbf{J})$ , and feed  $\Phi_{\ell m}^P$  and  $\Phi_{\ell m}^Q$  from Eqs. (1)–(3) along with Eq. (19) into Eq. (18), and invoke the addition theorem and orthonormality of the  $Y_{\ell m}$ 's to obtain

$$a_{j\ell}^Q = \frac{2j+1}{2\ell+1} K_{j\ell} a_{j\ell}^P. \quad (21)$$

Reading off  $a_{j\ell}^Q$  from Eq. (17), we immediately obtain Eq. (2) for  $a_{j\ell}^P$ .

We now find  $\Phi_{\ell m}^W(\hat{\mathbf{n}})$ . We evaluate the trace in Eq. (8) by writing Eq. (6) for  $A$ , and multiply by  $B$  to obtain

$$\langle \Phi_A^W(\hat{\mathbf{n}}) \Phi_B^W(\hat{\mathbf{n}}) \rangle_{\hat{\mathbf{n}}} = \langle AB \rangle_{\text{qm}} = \langle \Phi_A^P(\hat{\mathbf{n}}) \Phi_B^Q(\hat{\mathbf{n}}) \rangle_{\hat{\mathbf{n}}}. \quad (22)$$

Naturally, we can switch the roles of P and Q on the right. If we take  $\mathcal{Y}_{\ell m}$  for  $A$  and  $\mathcal{Y}_{\ell' m'}$  for  $B$ , we find Eq. (4) for  $a_{j\ell}^W$  when  $\ell = \ell'$ ,  $m = m'$ . More generally,

$$\langle \mathcal{Y}_{\ell m}(\mathbf{J}) \mathcal{Y}_{\ell' m'}^\dagger(\mathbf{J}) \rangle_{\text{qm}} = \frac{1}{4\pi} (a_{j\ell}^W)^2 \delta_{\ell\ell'} \delta_{mm'}, \quad (23)$$

which is an operator orthogonality relation. In particular,  $a_{j1}^W = [j(j+1)]^{1/2}$ , so that for  $\mathbf{J}$  itself, the Weyl symbol is  $\sqrt{j(j+1)} \hat{\mathbf{n}}$ . This is a direct deduction of exactly what is indicated for the “classical” vector corresponding to  $\mathbf{J}$  by a large number of indirect quantum mechanical arguments. Further, for fixed  $\ell$ , we find that

$$\Phi_{\ell m}^W(\mathbf{J}) \approx \left(1 + \frac{\ell}{2j} + \dots\right) j^\ell Y_{\ell m}(\hat{\mathbf{n}}), \quad (j \rightarrow \infty). \quad (24)$$

The above formulas allow us to obtain a miscellany of other results. First, we can invert Eq. (18) to go from the Q symbol to the P symbol:

$$\Phi_A^P(\hat{\mathbf{n}}) = \frac{1}{4\pi} \int d\hat{\mathbf{n}}' G(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \Phi_A^Q(\hat{\mathbf{n}}'), \quad (25)$$

where

$$G(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = \sum_{\ell=0}^{2j} (2\ell+1) \frac{(2j-\ell)!(2j+\ell+1)!}{(2j)!(2j+1)!} P_\ell(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'). \quad (26)$$

The coefficients in this formula are essentially  $1/K_{j\ell}$ , so it is somewhat formal, and it would be interesting to obtain a result in closed form.

Second, we can find asymptotic relations between the different symbols in the classical limit, by which we mean that we let  $j \rightarrow \infty$  keeping fixed the form of the operator  $A$  as a function of  $\mathbf{J}$ . To get  $\Phi_A^P$  from  $\Phi_A^Q$ , e.g., we take Eqs. (2) and (3), and expand the ratio in powers of  $1/j$  to get

$$\frac{a_{j\ell}^P}{a_{j\ell}^Q} = 1 + \frac{\ell(\ell+1)}{2j} + \frac{\ell(\ell+1)(\ell(\ell+1)-2)}{8j^2} + \dots \quad (27)$$

Now,  $\Phi_{\ell m}^P = a_{j\ell}^P \Phi_{\ell m}^Q / a_{j\ell}^Q$ , and  $\ell(\ell+1)Y_{\ell m}(\hat{\mathbf{n}}) = \mathcal{L}^2 Y_{\ell m}(\hat{\mathbf{n}})$ , where  $\mathcal{L} = -i(\hat{\mathbf{n}} \times \nabla_{\hat{\mathbf{n}}})$  is the angular momentum operator (on phase space, and *not* the quantum mechanical Hilbert space). When written in terms of  $\mathcal{L}^2$ , the ratio  $a_{j\ell}^P/a_{j\ell}^Q$  has no  $\ell$  dependence, and so constitutes an operator which applied to  $\Phi_A^Q$  will yield  $\Phi_A^P$  for *any*  $A$ . Up to order  $j^{-3}$ , we find [11]

$$\Phi_A^P(\hat{\mathbf{n}}) = \left(1 + \frac{\mathcal{L}^2}{2j} + \frac{1}{8j^2} \mathcal{L}^2(\mathcal{L}^2 - 2) + \frac{1}{48j^3} \mathcal{L}^2(\mathcal{L}^2 - 2)(\mathcal{L}^2 - 3) + \dots\right) \Phi_A^Q(\hat{\mathbf{n}}). \quad (28)$$

Similarly,

$$\Phi_A^P(\hat{\mathbf{n}}) = \left(1 + \frac{\mathcal{L}^2}{4j} + \frac{1}{32j^2} \mathcal{L}^2(\mathcal{L}^2 - 4) + \frac{1}{384j^3} \mathcal{L}^2(\mathcal{L}^4 - 8\mathcal{L}^2 + 24) + \dots\right) \Phi_A^W(\hat{\mathbf{n}}). \quad (29)$$

We can also find the Stratonovich-Weyl kernel  $\Delta^j(\hat{\mathbf{n}})$  [5, 8], which is an operator valued function on the unit sphere with the properties that

$$\Phi_A^W(\hat{\mathbf{n}}) = \langle A \Delta^j(\hat{\mathbf{n}}) \rangle_{\text{qm}}, \quad A = \langle \Phi_A^W(\hat{\mathbf{n}}) \Delta^j(\hat{\mathbf{n}}) \rangle_{\hat{\mathbf{n}}}. \quad (30)$$

It is straightforward to show that

$$\Delta^j = 4\pi \sum_{\ell, m} Y_{\ell m}^*(\hat{\mathbf{n}}) \mathcal{Y}_{\ell m}(\mathbf{J}) / a_{j\ell}^W. \quad (31)$$

In Ref. [8] instead,  $\Delta^j$  is given as a matrix in the  $|j, m\rangle$  basis. It is easily seen that  $\Delta^j$  must obey the rules,

$$\langle \Delta^j(\hat{\mathbf{n}}) \rangle_{\hat{\mathbf{n}}} = 1, \quad \langle \Delta^j(\hat{\mathbf{n}}) \Delta^j(\hat{\mathbf{n}}') \rangle_{\text{qm}} = 4\pi I^j(\hat{\mathbf{n}}, \hat{\mathbf{n}}'). \quad (32)$$

Here,  $I^j(\hat{\mathbf{n}}, \hat{\mathbf{n}}')$  is the identity kernel on the space of harmonic functions on the unit sphere that are of order  $2j$  or less. That is,

$$\int d\hat{\mathbf{n}}' I^j(\hat{\mathbf{n}}, \hat{\mathbf{n}}') f(\hat{\mathbf{n}}') = f(\hat{\mathbf{n}}) \quad (33)$$

for any function  $f(\hat{\mathbf{n}})$  whose spherical harmonic expansion contains no terms with  $\ell > 2j$ . The first property of  $\Delta^j$  follows almost trivially from Eq. (31), and the second from Eq. (23).

We turn, finally, to determining the asymptotic ( $j \rightarrow \infty$ ) behavior of the Moyal product, i.e., the Weyl symbol for a product of operators  $AB$  in terms of the symbols for the individual operators  $A$  and  $B$ . Using the  $\Delta^j$  mapping, one finds that [8]

$$\Phi_{AB}^W(\hat{\mathbf{n}}_1) = \langle M_j(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3) \Phi_A^W(\hat{\mathbf{n}}_2) \Phi_B^W(\hat{\mathbf{n}}_3) \rangle_{\hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3}; \quad (34)$$

$$M_j(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3) = \langle \Delta^j(\hat{\mathbf{n}}_1) \Delta^j(\hat{\mathbf{n}}_2) \Delta^j(\hat{\mathbf{n}}_3) \rangle_{\text{qm}}, \quad (35)$$

As  $j \rightarrow \infty$ , the kernel  $M_j$  is very sharply peaked when  $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2 = \hat{\mathbf{n}}_3$  and nearly zero otherwise. Progressively higher moments of  $M_j$  are of higher order in  $1/j$ , suggesting a Taylor expansion of  $\Phi_A^W$  and  $\Phi_B^W$ . Now,

$$\Phi_A^W(\hat{\mathbf{n}}_2) = e^{i\theta_{12}\mathcal{L}_{\perp,12}} \Phi_A^W(\hat{\mathbf{n}}_1), \quad (36)$$

where  $\theta_{12}$  is the angle between  $\hat{\mathbf{n}}_2$  and  $\hat{\mathbf{n}}_1$ , and  $\mathcal{L}_{\perp,12}$  is the component of  $\mathcal{L}$  orthogonal to both  $\hat{\mathbf{n}}_1$  and  $\hat{\mathbf{n}}_2$ . We have

$$\begin{aligned} \theta_{12}\mathcal{L}_{\perp,12} &= \frac{\theta_{12}}{\sin\theta_{12}} (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \cdot \mathcal{L} \\ \frac{\theta_{12}}{\sin\theta_{12}} &= \left[ 1 + \frac{1}{6} (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)^2 + \frac{3}{40} (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)^4 + \dots \right]. \end{aligned} \quad (37)$$

We thus generate a series in powers of gradients of  $\Phi_A^W$  and  $\Phi_B^W$ . Omitting the arguments of  $M_j$ , we have

$$\langle M_j f(\hat{\mathbf{n}}_i) \rangle = f(\hat{\mathbf{n}}_i), \quad (i = 2, 3), \quad (38)$$



for any  $f(\hat{\mathbf{n}})$ . Hence, beyond the leading term, all terms in the Moyal product will entail gradients of *both*  $\Phi_A^W$  and  $\Phi_B^W$ . The first non trivial term (of order  $j^{-1}$ ) arises from the moment

$$\langle M_j n_{2\alpha} n_{3\beta} \rangle = \frac{a_{j2}^W}{(a_{j1}^W)^2} n_{1\alpha} n_{1\beta} + \frac{i}{2a_{j1}^W} \epsilon_{\alpha\beta\gamma} n_{1\gamma} + c \delta_{\alpha\beta}, \quad (39)$$

where  $c = (a_{j2}^W - j(j+1))/3(a_{j1}^W)^2 = O(j^{-2})$ . The first term contributes nought, and the third only at order  $1/j^2$ . The central term leads to

$$\frac{i}{2j_c} \hat{\mathbf{n}} \cdot (\mathcal{L}\Phi_A^W \times \mathcal{L}\Phi_B^W) = \frac{i}{2j_c} \hat{\mathbf{n}} \cdot (\nabla_{\hat{\mathbf{n}}}\Phi_A^W \times \nabla_{\hat{\mathbf{n}}}\Phi_B^W), \quad (40)$$

where  $j_c \equiv \sqrt{j(j+1)}$  and we have written  $\hat{\mathbf{n}}$  for  $\hat{\mathbf{n}}_1$ .

We can find moments of higher order tensors such as  $n_{2\alpha} n_{2\beta} n_{3\gamma}$  in the same way. Odd order tensors do not lead to any corrections, and only even orders need to be considered. However, even finding all fourth order moments and the corresponding part of the Moyal product of order  $j^{-2}$  is laborious and quite daunting. So, in summary, what we have found up to now is that

$$\Phi_{AB}^W(\hat{\mathbf{n}}) = \Phi_A^W(\hat{\mathbf{n}})\Phi_B^W(\hat{\mathbf{n}}) + \frac{i}{2j_c} \hat{\mathbf{n}} \cdot (\nabla_{\hat{\mathbf{n}}}\Phi_A^W \times \nabla_{\hat{\mathbf{n}}}\Phi_B^W) + O(j^{-2}). \quad (41)$$

The term of order  $1/j$  may be written more physically in terms of the Weyl symbol for the commutator  $[A, B]$ . If we define  $j_c \hat{\mathbf{n}} = \mathbf{j}_c$  as the classical angular momentum vector, we may cast it as

$$\Phi_{[A,B]}^W(\mathbf{j}_c) \approx i \mathbf{j}_c \cdot \left( \frac{\partial \Phi_A^W(\mathbf{j}_c)}{\partial \mathbf{j}_c} \times \frac{\partial \Phi_B^W(\mathbf{j}_c)}{\partial \mathbf{j}_c} \right). \quad (42)$$

But the right hand is precisely the Poisson bracket of two functions of  $\mathbf{j}_c$ , given that the fundamental Poisson brackets are  $\{j_{c\alpha}, j_{c\beta}\}_{\text{PB}} = \epsilon_{\alpha\beta\gamma} j_{c\gamma}$ . Hence we have shown that to leading order as  $j \rightarrow \infty$ ,

$$\Phi_{[A,B]}^W \approx i \{\Phi_A^W, \Phi_B^W\}_{\text{PB}}. \quad (43)$$

## Acknowledgments

This work was supported by the NSF via grant numbers PHY-0854896 and DGE-0801685 (NSF-IGERT program).

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- [11] The same result may be found more cumbrously by performing a gradient expansion on Eq. (18), and inverting the series.